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SOME EQUATIONS OF MECHANICS POSSESSING FIRST INTEGRALS†

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The question of the structure of systems of differential equations, which arise in certain problems of mechanics and allow of first integrals is discussed. © 2003 Elsevier Science Ltd. All rights reserved.

As is well known, the existence of first integrals in a system of differential equations makes a considerable impression on its form (see, for example, [1]). Thus [2], the existence linear system of a quadratic first integral, when natural conditions of non-degeneracy are satisfied, predetermines its Hamiltonian structure. The connection between the existence of first integrals in the system and the existence in it of a Hamiltonian or Poisson structure can also be traced in other examples (see, for example, [3]). This connection anticipates one of the approaches to the general problem, formulated previously in [4], of finding the Hamiltonian structure of systems of ordinary differential equations.

1. EULER-TYPE EQUATIONS

The special properties of Hamiltonian systems distinguish them from the general set of systems of differential equations. Hence, even information on the existence of a Hamiltonian structure, possibly without its explicit detection, is of considerable interest (see, for example, [5–9]). It has been known even from Jacobi's time that in certain cases it is possible to make progress in solving the problem of the existence of a Hamiltonian or Poisson structure for systems of ordinary differential equations with a sufficient number of first integrals. Thus, if an autonomous system of n differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n \tag{1.1}$$

allows of $n-1$, in general, independent first integrals $j_1, j_2, \ldots, j_{n-1}$, it can be represented in the form (see, for example, [2, Chapter 11], [10] and also [11])

$$\dot{x}_i = \alpha(\mathbf{x})\frac{\partial(x_i, J_1, J_2, \ldots, J_{n-1})}{\partial(x_1, x_2, \ldots, x_n)}, \quad i = 1, 2, \ldots, n \tag{1.2}$$

where $\alpha(\mathbf{x})$ is a certain function. The region $\alpha(\mathbf{x}) = 0$ consists of fixed points. Hence, both it and the regions

$$\Sigma^- = \{\mathbf{x}: \alpha(\mathbf{x}) < 0\}, \quad \Sigma^+ = \{\mathbf{x}: \alpha(\mathbf{x}) > 0\}$$

are invariant under the action of a phase flow, and in each of them, by changing the time $d\tau = dt\alpha(\mathbf{x})$, system (1.2) can be converted to the form

$$x_i' = \frac{\partial(x_i, J_1, J_2, \ldots, J_{n-1})}{\partial(x_1, x_2, \ldots, x_n)}, \quad i = 1, 2, \ldots, n \tag{1.3}$$

The limitation of the phase flow of system (1.2) on the non-singular combined level of $n-2$ of these integrals, let us say, on the surface

$$I = (\mathbf{x}: J_2 = j_2, J_3 = j_3, \ldots, J_{n-1} = j_{n-1}) \tag{1.4}$$

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is described, apart from the time change, by a system of Hamilton equations with one degree of freedom with Hamilton function $J = J_{1|I}$.

Naturally, in the general case, systems of the form (1.1) do not possess that number of first integrals which enable them to be represented in the form (1.2). Nevertheless, even in this case it is possible to draw certain conclusions regarding the structure of system (1.1). Thus, for example, when $n = 3$ the following assertion holds.

Assertion 1. If system (1.1) possesses at least one differentiable first integral $J = J(\mathbf{x})$ or, in other words, the following relation is satisfied

$$(\partial J/\partial \mathbf{x}, \mathbf{f}(\mathbf{x})) = 0 \tag{1.5}$$

then system (1.1) can be represented in the form

$$\dot{\mathbf{x}} = \alpha(\mathbf{x})\mathbf{e}(\mathbf{x}) + \beta(\mathbf{x})\mathbf{e}(\mathbf{x}) \times \mathbf{g}(\mathbf{x}), \quad \mathbf{g} = \partial J/\partial \mathbf{x} \tag{1.6}$$

where $\mathbf{e}(\mathbf{x})$ is an arbitrary three-dimensional vector, orthogonal to the vector $\mathbf{g}$.

In particular, if $(g_1 - g_2)(g_2 - g_3)(g_3 - g_1) \neq 0$, we can chose the vector $\mathbf{e} = (g_3 - g_2, g_1 - g_3, g_2 - g_1)$ as this vector. In this case

$$(\mathbf{e} \times \mathbf{g})_1 = g_3(g_1 - g_3) - g_2(g_2 - g_1) \ (123) \tag{1.7}$$

If $J \neq J(x_1^2 + x_2^2 + x_3^2)$ in a certain region $\mathcal{U}$, we can choose

$$\mathbf{e} = \mathbf{x} \times \mathbf{g}(\mathbf{x}) \tag{1.8}$$

as this vector in this region.

Proof. We will consider relation (1.5) as an equation in the components of the vector $\mathbf{f}$ for a fixed value of $\mathbf{x}$. If $J_c = \{\mathbf{z}: J(\mathbf{z}) = c\}$ is the level of the first integral, containing the point $\mathbf{x}$, the linear homogeneous equation (1.5) defines a set of vectors, collinear with the plane which touches the surface J_c at the point $\mathbf{x}$. Then, by the properties of a mixed product, if a certain non-zero vector vector $\mathbf{g}$ and $\mathbf{e} \times \mathbf{g}$ define a general two-parameter solution of Eq. (1.5) having the form

$$\mathbf{f}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{e}(\mathbf{x}) + \beta(\mathbf{x})\mathbf{e}(\mathbf{x}) \times \mathbf{g}(\mathbf{x}) \tag{1.9}$$

whence the correctness of the first proposition follows.

The proof of the second and third propositions is based on the fact that for any vector $\mathbf{h}(\mathbf{x})$, not collinear with the vector $\mathbf{g}(\mathbf{x})$, the vector $\mathbf{h} \times \mathbf{g}$ is orthogonal to both of them and is collinear with this plane. In the second proposition the vector $\mathbf{h}$ is constant and has the form $\mathbf{h} = (1, 1, 1)$, and in the third proposition $\mathbf{h} = (x_1, x_2, x_3)$.

In the last case if $\beta(\mathbf{x}) = 0$, after changing the time $d\tau = \alpha(x)dt$, Eqs (1.6) become Euler's equations (the prime denotes a derivative with respect to τ)

$$\mathbf{x}' = \mathbf{x} \times \mathbf{g} \tag{1.10}$$

which possess the well-known Poisson structure.

The existence of two independent first integrals J_1 and J_2 in the case when $n = 3$ enables us, in view of Jacobi's theorem, to represent Eqs (1.1) in the form

$$\dot{\mathbf{x}} = \alpha(\mathbf{x})(\partial J_1/\partial \mathbf{x}) \times (\partial J_2/\partial \mathbf{x}) \tag{1.11}$$

and to represent Eqs (1.3) in the form

$$\mathbf{x}' = (\partial J_1/\partial \mathbf{x}) \times (\partial J_2/\partial \mathbf{x}) \tag{1.12}$$

These equations, in particular, describe the motions of an Euler top for $J_1 = (\mathbf{x}, \mathbf{x})/2$ and of a balanced gyrostat for $J_1 = (\mathbf{x} + \mathbf{c}, \mathbf{x} + \mathbf{c})/2$.

Assertion 2. Suppose one of the integrals of Eqs (1.12), say, integral J_1, is quadratic and has the form

$$J_1 = 1/2(\mathbf{B}\mathbf{x}, \mathbf{x}), \quad \mathbf{B} = \mathbf{B}^T \tag{1.13}$$

where the matrix **B** is positive definite. Then, a non-degenerate linear homogeneous change of variables exists, which, in combination with the time change, reduces Eqs (1.12) to the form of Euler's equations (1.10).

Proof. Since the matrix **B** is positive definite, a matrix **C** exists such that $\mathbf{B} = \mathbf{C}^2$, where the determinant of the matrix **C** is non-zero. Then, the change of the variables

$$\mathbf{x}^* = \mathbf{C}\mathbf{x} \tag{1.14}$$

reduces the integrals J_1 and J_2 to the form

$$J_1^* = (\mathbf{x}^*, \mathbf{x}^*), \quad J^* = J_2(\mathbf{C}^{-1}\mathbf{x}) \tag{1.15}$$

By Jocobi's theorem, in the new variables, Eqs (1.3) can be represented in the form

$$\dot{\mathbf{x}}^* = \alpha^*(\mathbf{x}^*)\mathbf{x}^* \times \partial J^*/\partial \mathbf{x}^* \tag{1.16}$$

Changing the time and dropping the asterisks we finally obtain Euler's equations in the form (1.10).

Note that the matrix **C** is not uniquely defined.

For systems with quadratic definite defined first integral, relation (1.14) gives a global change of variables, which enables system (1.11) to be reduced to Euler's equations. If, in general, we drop the globality condition, the assertion proved can be strengthened considerably.

Assertion 3. Suppose one of the integrals of Eqs (1.12), say, integral J_1, reaches a strict minimum (maximum) at the point $\mathbf{x}_0$, and the determinant of the Hesse matrix of the function J_1 at the point $\mathbf{x}_0$ is non-zero. We then obtain and open and region $\mathcal{U}$, containing the point $\mathbf{x}_0$, such that Eqs (1.12) appear in the form of Euler's equations (1.10) in a certain system of coordinates after an appropriate time change.

Proof. By Morse's lemma ([12], see also [13, Chapter 6]) in a certain neighbourhood of the non-degenerate minimum (maximum) of the function J_1, we can choose coordinates such that in them $J_1 = -(x_1^2 + x_2^2 + x_3^2)/2$ (correspondingly $J_1 = -(x_1^2 + x_2^2 + x_3^2)/2$), in which case the proof of the assertion reduces to using Jacobi's theorem and changing the time.

Remark. Morse's lemma also enables the system to be converted to a certain similar canonical form in the neighbourhood of non-degenerate saddle points. However, the equations obtained are not encountered in applications as often as systems of Euler's equations. Besides, such "diversity of cases" is easily removed by considering equations in the set of complex numbers.

2. LINEAR SYSTEMS WHICH POSSESS FIRST INTEGRALS

We will now assume that system (1.1) is linear and has the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in R^3, \quad \mathbf{A} = \text{const} \tag{2.1}$$

Suppose the determinant of matrix **A** is equal to zero. Then, both the determinant of the transposed matrix $\mathbf{A}^T$ and one of its eigenvalues are also equal to zero. We will denote the eigenvector of the matrix $\mathbf{A}^T$, corresponding to this zero eigenvalue, by $\mathbf{g} = (g_1, g_2, g_3)$.

Assertion 4. If $\det A = 0$, system (2.1) can be represented in the form

$$\dot{\mathbf{x}} = \alpha \mathbf{x} \times g + \beta(\mathbf{x} \times \mathbf{g}) \times \mathbf{g}, \quad \alpha, \beta = \text{const} \tag{2.2}$$

Proof. It is well known that, under the conditions of the theorem, the function

$$J = (\mathbf{g}, \mathbf{x}) \tag{2.3}$$

is the first of Eqs (2.1): differentiating this function we have, by virtue of system (2.1)

$$d(\mathbf{g}, \mathbf{x})/dt = (\mathbf{g}, \mathbf{A}\mathbf{x}) = (\mathbf{A}^T\mathbf{g}, \mathbf{x}) = 0$$

Then, the conditions of Assertion 1 are satisfied and Eqs (2.1) can be represented in the form (1.6). To complete the proof it remains to note that the factor α and β are constant, by virtue of the linearity of system (2.1).

Since $\mathbf{g}$ is a constant eigenvector of the constant matrix $\mathbf{A}$, we will assume, without loss of generality, that $\alpha = 1$.

If $\beta = 0$, system (2.2) is a system of Euler's equations of the form (1.10) with Hamilton function (2.3).

Following the formulation of the problem, proposed previously in [2], we will now consider the case when Eqs (2.1) allow of a quadratic first integral of the form (1.13), where $\det\mathbf{B} \neq 0$.

Assertion 5. If system (2.1) possesses a non-degenerate quadratic integral (1.13), then $\det\mathbf{A} = 0$ and system (2.1) can be represented in the form

$$\dot{\mathbf{x}} = \mathbf{g} \times \mathbf{B}\mathbf{x} \tag{2.4}$$

where $\mathbf{g}$ is constant eigenvector of the matrix $\mathbf{A}^T$, corresponding to its zero eigenvalue.

Proof. Since (1.13) is a first integral, the equality

$$(\mathbf{B}\mathbf{x}, \mathbf{A}\mathbf{x}) = (\mathbf{A}^T\mathbf{B}\mathbf{x}, \mathbf{x}) = 0$$

is satisfied identically. The matrix $\mathbf{A}^T\mathbf{B}$ is then skew-symmetric, and its determinant is equal to zero, since the dimension of the matrix is odd. But the matrix $\mathbf{B}$ is non-degenerate and $\det\mathbf{B} \neq 0$. So, $\det\mathbf{A} = 0$ and $\det\mathbf{A}^T = 0$. Then, in view of the property of the corresponding zero eigenvalue of the eigenvector of the matrix $\mathbf{A}^T$ mentioned above, the function of the form (2.3) is a first integral of Eqs (2.1). Consequently, by virtue of Jacobi's theorem, the equations can be represented in the form (1.12). In this case, since the equations are linear, the time change turns out to be unnecessary – it can be compensated by choosing the eigenvector $\mathbf{g}$, defined, as we know, apart from a factor.

Note that we can give the search for the vector $\mathbf{g}$ a "more constructive form". Since the matrix $\mathbf{B}$ is non-degenerate, an inverse matrix $\mathbf{B}^{-1}$ exists by means of which we can represent Eqs (2.1) as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{B}^{-1})\mathbf{B}\mathbf{x}$$

The matrix $\mathbf{A}\mathbf{B}^{-1}$ is skew-symmetric, since for any vector $\mathbf{z}$ we have (compare [2])

$$(\mathbf{A}\mathbf{B}^{-1}\mathbf{z}, \mathbf{z}) = (\mathbf{A}\mathbf{x}, \mathbf{B}\mathbf{x}) \equiv 0 \quad (\mathbf{x} = \mathbf{B}^{-1}\mathbf{z})$$

Then the correspondence between the components of this matrix and the components of the vector $\mathbf{g}$ is established using the known isomorphism between 3×3 skew-symmetric matrices and three-dimensional vectors, having the form

$$g_1 = (AB^{-1})_{23} \ (123)$$

Assertion 6. Under the conditions of Assertion 5 suppose a symmetric matrix $\mathbf{C}$ exists such that

$$\mathbf{B}^2 = \mathbf{C} \tag{2.5}$$

Then, by the linear change of variables

$$\mathbf{x}' = \mathbf{C}\mathbf{x} \tag{2.6}$$

Eqs (2.1) can be reduced to a system of Euler's equations of the form (1.10).

Proof. Since $\det\mathbf{B} \neq 0$, we have $\det\mathbf{C} \neq 0$, and the inverse matrix $\mathbf{C}^{-1}$ exist. By the change of variables (2.6), Eqs (2.1) and integral (1.13) can be reduced to the form

$$\dot{\mathbf{x}}^* = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}\mathbf{x}^* = \mathbf{A}^*\mathbf{x}^* \tag{2.7}$$

and

$$J = (\mathbf{x}^*, \mathbf{x}^*) \tag{2.8}$$

respectively.

The function (2.8) is a first integral of Eqs (2.7). Discussions similar to those above show that the matrix $\mathbf{A}^*$ is skew-symmetric. Then, assuming

$$g_1^* = -(\mathbf{A}^*)_{23}\ (123)$$

and dropping the asterisks everywhere within this assertion, we can represent Eqs (2.1) in the form of Euler's equations

$$\dot{\mathbf{x}} = \mathbf{x}\times\mathbf{g} = \mathbf{x}\times\frac{\partial J}{\partial \mathbf{x}} \tag{2.9}$$

with Hamilton function

$$J = (\mathbf{g}, \mathbf{x}) \tag{2.10}$$

These results to a certain extent supplement the results obtained previously in [2].

3. SOME EQUATIONS WHICH ARISE IN RIGID BODY MECHANICS

When investigating a number of mechanical systems, the equations of motion can be represented in the the form

$$\dot{\mathbf{M}} = \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \mathbf{P} + \mathbf{Q}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\frac{\partial H}{\partial \mathbf{M}} + \kappa\mathbf{M}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \mathbf{M}, \boldsymbol{\gamma}\in R^3, \quad \kappa = 0, \pm 1 \tag{3.1}$$

Consider system (3.1) when

$$\mathbf{P} = \boldsymbol{\gamma}\times\partial H/\partial\boldsymbol{\gamma}, \quad \mathbf{Q} = 0 \tag{3.2}$$

Equations arise in particular when $\varkappa = 0$ in the problem of the motion of a rigid body around a fixed point in an axisymmetrical force field, and in the problem of the motion of a rigid body in an ideal incompressible fluid; when $\varkappa = 1$ they arise in the problem of the motion of a rigid body with an ellipsoidal cavity filled with an ideal liquid performing uniform vortical motion; when $\varkappa = -1$ they arise in the problem of the motion of a rigid body along the surface of constant negative curvature.

If the Hamilton function H is explicitly independent of time, it will be a first integral of the equations of motion. Moreover, the functions

$$J_1 = (\mathbf{M}, \boldsymbol{\gamma}), \quad J_2 = 1/2((\boldsymbol{\gamma}, \boldsymbol{\gamma}) + \kappa(\mathbf{M}, \mathbf{M})) \tag{3.3}$$

are first integrals of system (3.1), (3.2), which the Hamilton function H would not be.

The following assertion, the proof of which consists of a direct check, holds.

Assertion 7. System of equations (3.1), (3.2) can be represented in the form

$$\dot{\mathbf{M}} = \frac{\partial J_1}{\partial \boldsymbol{\gamma}}\times\frac{\partial H}{\partial \mathbf{M}} + \frac{\partial J_1}{\partial \mathbf{M}}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \frac{\partial J_1}{\partial \mathbf{M}}\times\frac{\partial H}{\partial \mathbf{M}} + \kappa\frac{\partial J_1}{\partial \boldsymbol{\gamma}}\times\frac{\partial H}{\partial \boldsymbol{\gamma}} \tag{3.4}$$

The limitation of system (3.1), (3.2) to a non-singular combined level of integrals J_1 and J_2 can be described by a canonical system of Hamilton equations with two degrees of freedom.

We will now consider system of equations (3.1) when

$$\mathbf{P} = 0, \quad \mathbf{Q} = \mathbf{Q}(\mathbf{M}, \boldsymbol{\gamma}) \tag{3.5}$$

where the generalized moment $\mathbf{Q}$ is a certain fairly continuous functions of its arguments.

Assertion 8. If system (3.1), (3.5) allows of first integrals H and J_1, it first subsystem can be represented in the form

$$\dot{\mathbf{M}} = (\mathbf{M} + \alpha\boldsymbol{\gamma})\times\frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \alpha = \alpha(\mathbf{M}, \boldsymbol{\gamma}) \tag{3.6}$$

Proof. Since the functions H and $\mathcal{T}_1$ are first integrals, the components of the vector $\mathbf{Q}$ satisfy the relations

$$\left(\frac{\partial H}{\partial \mathbf{M}}, \mathbf{Q}\right) + \left(\boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}}, \frac{\partial H}{\partial \boldsymbol{\gamma}}\right) = 0, \quad (\boldsymbol{\gamma}, \mathbf{Q}) = 0 \tag{3.7}$$

or

$$\left(\frac{\partial H}{\partial \mathbf{M}}, \mathbf{Q}^*\right) = 0, \quad (\boldsymbol{\gamma}, \mathbf{Q}^*) = 0, \quad \mathbf{Q}^* = \mathbf{Q} + \frac{\partial H}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma} \tag{3.8}$$

Then, by virtue of relations (3.7), $\mathbf{Q}^* = \alpha\gamma \times \partial H/\partial \mathbf{M}$, which, taking the last relation of (3.8) into account, proves the required result.

When $\varkappa = 0$ and $\alpha = 0$ Eqs (3.1) and (3.5) are the Euler–Poisson equations. This relation between the existence of an integral of the energy-integral type and the presence in the system or a Poisson structure was investigated previously in [3]. We will present the corresponding results as an example.

Example 1. *The motion of a body with a fixed point in a flow of particles.* Consider the motion of a rigid body about a fixed point G under the action of a uniform flow of particles, directed along the unit vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, fixed in absolute space. We will assume that the particles collide absolutely elastically with the body, ρ is the density of the particle flux, V is the flow velocity, $S(\gamma)$ is the mid-section area of the body (i.e. the area of the projection of the body onto the plane perpendicular to the flow), and $\mathbf{c} = (c_1, c_2, c_3)$ is the vector connecting the fixed point with an arbitrary point of the straight line collinear with the vector $\boldsymbol{\gamma}$ and passing through the centroid of the mid-section of the body $\mathcal{S}(\gamma)$. The vector $\mathbf{c}$ in general depends on $\boldsymbol{\gamma}$. If $\mathbf{I}$ is the inertia tensor of the body with respect to the fixed point and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the vector of the angular velocity of the body, then, when the flow velocity considerably greater than the linear velocities of points of the body, the equations of motion have the form [3]

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + f\boldsymbol{\gamma} \times \mathbf{c}(\boldsymbol{\gamma})S(\boldsymbol{\gamma}), \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad f = \rho V^2 \tag{3.9}$$

It was pointed out in [3] that if a function $U = U(\gamma)$ exist such that

$$\mathbf{c}(\boldsymbol{\gamma})\mathbf{S}(\boldsymbol{\gamma}) = \partial U/\partial \gamma \tag{3.10}$$

Eqs (3.9) are the Poincaré–Chetayev equations. However, in general, condition (3.10) does not occur. Thus, for a body bounded by an ellipsoidal with semi-axes b_1, b_2, and b_3, collinear with the axes of the system of coordinates connected with the body, the mid-section area S is given by the relation

$$S(\gamma) = \pi b_1 b_2 b_3 (\gamma_1^2/b_1^2 + \gamma_2^2/b_2^2 + \gamma_3^2/b_3^2)^{1/2} \tag{3.11}$$

We can take as the vector $\mathbf{c}$, as in any other case, when the body surface is centrally symmetric, the vector connecting the fixed point and the centre of the ellipsoidal surface. This vector is constant in the system of coordinates connected with the body, in view of which condition (3.10), when all the semi-axes b_i are different, turns out to be unsatisfied. Nevertheless, if the body is bounded by a surface of revolution with axis passing through the fixed point, where $b_1 = b_2 = b$ and $\mathbf{c} = (0, 0, c_3)$, then, by virtue of the fact that the relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ is satisfied, the potential U exists and be written in the form of an elliptic quadrature

$$U(\gamma_3) = \pi b b_3 c_3 \int_0^{\gamma_3} \left(1 + \frac{b^2 - b_3^2}{b_3^2} u^2\right)^{1/2} du \tag{3.12}$$

If the tensor $\mathbf{I} = (I_1, I_2, I_3)$ is symmetric and $I_1 = I_2$, the equations of motion allow of an additional integral $J_3 = I_3\omega_3$ and turns out to be completely integrable.

As is well known to integrate Eqs (3.1) and (3.2) one first integral is missing in the general case. Classical cases of the existence of such a first integral (see, for example, [4]), and also the case recently obtained in [14], are well known in mechanics.

Consider system (3.1) when

$$\mathbf{P} = \boldsymbol{\gamma} \times \partial H/\partial \boldsymbol{\gamma}, \quad \mathbf{Q} \neq 0 \tag{3.13}$$

It is asked, for what conditions, imposed on the components of the vector **Q**, will Eqs (3.1) and (3.13) retain the additional integral?

Suppose, for example, that the additional integral has the form

$$J_3 = 1/2(\mathbf{M}, \mathbf{M}) + U(\gamma) \tag{3.14}$$

as occurs, say, in the Clebsch case.

Assertion 9. If, in addition to integral J_1, Eqs (3.1) and (3.13) allow of an integral of the form (3.14), then

$$\mathbf{Q} = \alpha\mathbf{M} \times \boldsymbol{\gamma}, \quad \alpha = \alpha(\mathbf{M}, \boldsymbol{\gamma}) \tag{3.15}$$

Proof. Differentiating J_1 and J_3 we have, by virtue of system (3.1), (3,13)

$$(\mathbf{M}, \mathbf{Q}) = 0, \quad (\boldsymbol{\gamma}, \mathbf{Q}) = 0 \tag{3.16}$$

whence we also obtain relation (3.15). It is interesting to note that if the generalized moment **Q** has the form (3.15), Eqs (3.14) also allow of integral J_2.

Example 2. The motion of a body with a fixed point in a magnetic field. The problem of the motion of a rigid body in a constant uniform magnetic field of strength γ, on the assumption that the body is made of a material which becomes magnetized on rotation (the Barnett–London effect), was considered in [4] and also in [15, 16]. If we neglect the de Haas–Einstein effect which dual to Barnett–London effect and consists of the fact that, on magnetization, a body of such a material begins to rotate, then according to the papers mentioned above, the equations of motion can be represented in the form

$$\dot{\mathbf{M}} = \mathbf{M} \times \mathbf{I}^{-3}\mathbf{M} + \mathbf{A}\mathbf{M} \times \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \mathbf{I}^{-1}\mathbf{M}$$

Then, if $\mathbf{A} = \alpha\mathbf{E}$, where $\alpha = \text{const}$, these equations belong to the class of Eqs (3.14) and (3.15).

4. THE HAMILTONIAN STRUCTURE OF LINEAR DIFFERENTIAL EQUATIONS

The question of the Hamiltonian structure of linear ordinary differential equations, which allow of a quadratic first integral, was investigated in [2]. As is well known, linear differential equations can always be integrated in explicit form, but in general the first integrals are rational. It turns out that in a number of cases one can make a non-linear change of variables which preserves the linearity of the differential equations and reduces the corresponding integrals to quadratic form.

We will consider the simplest example. Suppose

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 \tag{4.1}$$

These equations are completely integrable, and they allow of two non-autonomous first integrals

$$F_i(x_i) = e^{-\lambda_i t} x_i, \quad i = 1, 2 \tag{4.2}$$

from which we can construct one first integral, explicitly independent of time, having the form

$$F = x_1^{\lambda_2} / x_2^{\lambda_1} \tag{4.3}$$

in the region $\mathscr{X} = \{\mathbf{x}: x_1 > 0, x_2 > 0\}$; the remaining regions are considered similarly.

In the region $\mathscr{X}$ we make the change of variables

$$X_i = x_i^{\mu_i}, \quad i = 1, 2$$

which only has singularities on the boundary of the region $\mathscr{X}$. Then

$$\dot{X}_i = \mu_i x_i^{\mu_i - 1} \dot{x}_1 = (\mu_i \lambda_i) X_i, \quad i = 1, 2$$

Assuming

$$\lambda_1\mu_1 = 1, \quad \lambda_2\mu_2 = -1$$

we can reduce the equations of motion to the form

$$\dot{X}_1 = X_1 = \partial H/\partial X_2, \quad \dot{X}_2 = -X_2 = -\partial H/\partial X_1, \quad H(X_1, X_2) = X_1X_2$$

We will consider the system

$$\dot{x}_1 = \lambda x_1 + x_2, \quad \dot{x}_2 = \lambda x_2 \tag{4.4}$$

As is well known, the general solution of this system has the form

$$x_1 = (C_2t + C_1)e^{\lambda t}, \quad x_2 = C_2e^{\lambda t}$$

whence the time-independent first integral can be written as

$$F = \lambda x_2 e^{-\lambda x_1/x_2}$$

Then, by the change of variables

$$X_1 = e^{-\lambda x_1/x_2}, \quad X_2 = x_2$$

Eqs (4.4) can be reduced to the form

$$\dot{X}_1 = \lambda X_1 = \frac{\partial H}{\partial X_2}, \quad \dot{X}_2 = -\lambda X_2 = -\frac{\partial H}{\partial X_1}, \quad H = \lambda X_1X_2$$

Hence, if the linear system in canonical form is formed by two-dimensional blocks of the form (4.1) or (4.4), the changes described above lead to a Hamiltonian form everywhere with the exception of the coordinate axes.

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